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Note

On the characterization of trees with signed edge domination numbers 1, 2, 3, or 4[☆]

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Abstract

Let $G = (V, E)$ be a simple graph. For an edge e of G, the closed edge-neighbourhood of e is the set $N[e] = \{e' \in E | e' \text{ is adjacent to } e\} \cup \{e\}$. A function $f : E \to \{1, -1\}$ is called *a signed edge domination function* (SEDF) of G if $\sum_{e' \in N[e]} f(e') \geq 1$ for every edge e of G. *The signed edge domination number* of G is defined as $\gamma_s'(G) = \min\{\sum_{e \in E} f(e) | f \text{ is an SEDF of } G\}$. In this paper, we characterize all trees T with signed edge domination numbers 1, 2, 3, or 4.

Keywords: Graph; Signed edge domination function; Signed edge domination number

1. Introduction

Let $G = (V, E)$ be a simple graph. For a vertex u of G, let $N_G(u)$ denote the open neighbourhood of u in G. The degree of u is denoted by $d_G(u)$. For a subset S of V(or E), let $G[S]$ be the subgraph of G induced by S and $G - S$ be the subgraph induced by $V - S$(or $E - S$). For an edge e of G, the open edge-neighbourhood of e is the set $N_G(e) = \{e' \in E | e' \text{ is adjacent to } e\}$ and the closed edge-neighbourhood of e is the set $N_G[e] = N_G(e) \cup \{e\}$. Let $\lfloor x \rfloor$ be the integer part of a nonnegative real number x.

A function $f : E \to \{+1, -1\}$ is called *a signed edge domination function* (SEDF) of G if $\sum_{e' \in N_G[e]} f(e') \geq 1$ for every edge e of G. *The signed edge domination number* of G is defined as $\gamma_s'(G) = \min\{\sum_{e \in E} f(e) | f \text{ is an SEDF of } G\}$. Denote $f(G) = \sum_{e \in E} f(e)$.

Xu [2] gave a lower bound for signed edge domination numbers of nontrivial trees.

Proposition 1 *([2]). If T be a nontrivial tree, then $\gamma_s'(T) \geq 1$.*

For other results on the signed edge domination number, the readers may refer to the survey papers of Xu [1,2].

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2. Main results

Let $T = (V, E)$ be a nontrivial tree. For a matching M of T, let D_M denote the subgraph of T induced by the edge set of those edges whose both ends are adjacent to M. If we contract each edge of M to a single vertex and discard edges which are not in D_M, then we have a forest T_M. It follows that $|E(D_M)| \leq |M| - 1$. Moreover, equality holds if and only if T_M is a tree.

For convenience, let $f[e]$ denote the sum of a function $f : E \to \{+1, -1\}$ on the edge e and its adjacent edges. Write $E_1 = \{e \in E | f(e) = 1\}$, $E_2 = \{e \in E | f(e) = -1\}$. For a vertex u, denote $d_f^*(u) = d_{G_1}(u) - d_{G_2}(u)$, where $G_i = G[E_i]$, $i = 1, 2$.

Lemma 1. *Let M be a maximal matching of a nontrivial tree T. If f is an SEDF of T, then $f(T) \geq |M| - f(D_M)$ and when $f(T) \leq 2$, $f(e) = 1$ for every non-pendant edge e.*

Proof. Since M is a maximal matching of T, every edge of T is either in M or adjacent to an edge of M. It follows that $\sum_{e \in M} f[e] = f(T) + f(D_M)$. On the other hand, $\sum_{e \in M} f[e] \geq |M|$. It follows that:

$$f(T) \geq |M| - f(D_M)$$

which is at least 1 by the above discussion.

If there exists an edge e of D_M such that $f(e) = -1$, then $f(D_M) \leq |M| - 3$. So when $f(T) \leq 2$, $f(e) = 1$ for any edge e of D_M. On the other hand, for any non-pendant edge e of $T - D_M$, we can choose a maximal matching M' of T such that e is an edge of $D_{M'}$. It follows that if $f(T) \leq 2$, then every non-pendant edge has f-value 1. □

Theorem 1. *Let T be a tree. Then $\gamma_s'(T) = 1$ if and only if every vertex of T has odd degree and for every non-pendant vertex v, there exist at least $\lfloor \frac{1}{2} d_T(v) \rfloor$ pendant edges incident with v.*

Proof. Sufficiency. Define a function $f : E(T) \to \{+1, -1\}$ as follows:

For every non-pendant vertex v of T, let the f-value of $\lfloor \frac{1}{2} d_T(v) \rfloor$ pendant edges incident with v be -1 and of other edges incident with v be 1.

Clearly, f is an SEDF of T with $f(T) = 1$. Since $\gamma_s'(T) \geq 1$ for every nontrivial tree T, $\gamma_s'(T) = 1$.

Necessity. Let f be an SEDF of T with $f(T) = 1$ and M be a maximal matching of T. Then by Lemma 1 and $|E(D_M)| \leq |M| - 1$, $f(e) = 1$ for every non-pendant edge e of T and $f(D_M) = |M| - 1$. Since $f(T) = \sum_{e \in M} f[e] - f(D_M)$, it follows that $f[e] = 1$ for every edge e of M. On the other hand, for any edge e of $T - M$, we can choose a maximal matching M' of T to contain e. Thus $f[e] = 1$ for every edge e of T.

Suppose that T has a vertex v not incident with any pendant edge. Then we can choose a maximal matching M of T that does not include any edge incident with v. In such a case, T_M has at least two components, and so $|D_M| < |M| - 1$, a contradiction.

Hence, we have shown that every non-pendant edge has f-value 1 and every vertex is incident with at least one pendant edge e, which by above has $f[e] = 1$. It follows that for every non-pendant vertex v, we have $d_f^*(v) = 1$: that is, it has odd degree and $\lfloor \frac{1}{2} d_T(v) \rfloor$ of its incident edges are pendant edges of f-value -1. □

Theorem 2. *Let T be a tree. Then $\gamma_s'(T) = 2$ if and only if T has exactly one even-degree vertex, and for every non-pendant vertex v, there exist at least $\lfloor \frac{1}{2}(d_T(v) - 1) \rfloor$ pendant edges incident with v.*

Proof. Sufficiency. Define a function $f : E(T) \to \{+1, -1\}$ as follows:

For every non-pendant vertex v of T, let the f-value of $\lfloor \frac{1}{2}(d_T(v) - 1) \rfloor$ pendant edges incident with v be -1 and of other edges incident with v be 1.

Clearly, f is an SEDF of T with $f(T) = 2$. By Theorem 1, $\gamma_s'(T) \geq 2$. Therefore $\gamma_s'(T) = 2$.

Necessity. Let f be an SEDF of T with $f(T) = 2$ and let M be any a maximal matching of T. Then by Lemma 1, $f(e) = 1$ for every non-pendant edge e of T and $f(D_M) \geq |M| - 2$.

Suppose that T has a vertex v not incident with any pendant edge and $d_T(v) \geq 3$. Then we can choose a maximal matching M that does not include any edge incident with v. In such a case, T_M has at least three components. So $|E(D_M)| \leq |M| - 3$, a contradiction. Therefore if T has a vertex v not incident with any pendant edge, then $d_T(v) = 2$.

Now we discuss two cases.

Case 1. If there exists a maximal matching M of T such that $f(D_M) = |M| - 2$, then $f[e] = 1$ for every edge e of M and there exists a 2-degree vertex v_0 such that uv_0 is not an edge of M for any u of $N_T(v_0)$. Clearly M also is a maximal matching of $T - v_0$. On the other hand, for any edge e of $T - v_0$ we can choose a maximal matching M' of T satisfying $f(D_{M'}) = |M'| - 2$ to contain e. Thus $f[e] = 1$ for every edge e of $T - v_0$.

Hence, we have shown that every non-pendant edge has f-value 1 and every vertex v with $d_T(v) \neq 2$ is incident with at least one pendant edge e, which by above, has $f[e] = 1$. It follows that for every vertex v with $d_T(v) \geq 3$, we have $d_f^*(v) = 1$: that is, it has odd degree and $\lfloor \frac{1}{2} d_T(v) \rfloor$ of its incident edges are pendant edges of f-value -1.

If there exists other 2-degree vertex v' not incident with any pendant edge, then for every vertex u of $N_T(v')$, by $f[uv'] = 1$, $f(uv') = 1$ and $d_f^*(v') = 2$, the degree of u is even and $d_f^*(u) = 0$. This implies that there are no pendant edges incident with u. But since $f(e) = 1$ for all non-pendant edges e, we must have $d_f^*(u) > 0$, a contradiction. Hence v_0 is the only even-degree vertex of T.

Case 2. If $f(D_M) = |M| - 1$ for any maximal matching M of T, then T cannot contain any 2-degree vertex not incident with any pendant edge, and by $\sum_{e \in M} f[e] = |M| + 1$, there exists an edge $e_0 = u_0 v_0$ of M such that $f[e_0] = 2$ and $f[e] = 1$ for any edge e of $M - e_0$. We affirm that one of u_0 and v_0 is even-degree. Otherwise $d_f^*(u_0)$ and $d_f^*(v_0)$ are both odd or both even. Hence $f[e_0] = d_f^*(u_0) + d_f^*(v_0) - f(e_0)$ is odd, a contradiction. Without loss of generality, we assume that v_0 is even-degree. Since v_0 must be incident with a pendant edge, and for every edge of T, we can choose a maximal matching of T to contain it, we assume that e_0 is a pendant edge. Clearly $M - e_0$ also is a maximal matching of $T - v_0$. On the other hand, for any edge e of $T - v_0$ we can choose a maximal matching M' of $T - v_0$ to contain it. Then $M' \cup e_0$ is a maximal matching of T, and by above, $f[e] = 1$ for every edge e of $T - E'$, where $E' = \{e \in E(T) | e = uv_0, u \in N_T(v_0)\}$. It follows that $f[e] = 2$ for every edge e of E'.

Hence, we have shown that every non-pendant edge has f-value 1 and every vertex is incident with at least one pendant edge e, which by above, has $f[e] = 1$ when e is an edge of $T - E'$ or $f[e] = 2$ when e is an edge of E'. It follows that $d_f^*(v_0) = 2$ and for every non-pendant vertex v of $T - v_0$, $d_f^*(v) = 1$: that is, T has exactly one even-degree vertex v_0 and for every vertex v of T, $\lfloor \frac{1}{2}(d_T(v) - 1) \rfloor$ of its incident edges are pendant edges of f-value -1. □

By the following lemma and induction on the number of vertices, we can obtain Theorems 3 and 4.

Lemma 2. *If f is an SEDF of a nontrivial tree T, then there exists at least one pendant edge e of T such that $f(e) = 1$.*

Proof. Assume, to the contrary, that for every edge e of T, e is a pendant edge, and $f(e) = -1$. Obviously T is not a star graph. Let U be the set of pendant vertices of T and $T' = T - U$. Then T' is a nontrivial tree and has a pendant vertex v. Then for a pendant edge e of T incident with v, we have $f[e] = d_f^*(v) \leq 0$, a contradiction. □

Theorem 3. *Let T be a tree. Then $\gamma_s'(T) = 3$ if and only if one of the following conditions is satisfied:*

(1) *Every vertex of T has odd degree; there exist two adjacent vertices v' and v'' with $d_T(v') \geq 3$ and $d_T(v'') \geq 3$ such that v' is incident with exactly $\lfloor \frac{1}{2}(d_T(v') - 2) \rfloor$ pendant edges and v'' is incident with at least $\lfloor \frac{1}{2}(d_T(v'') - 2) \rfloor$ pendant edges; for every non-pendant vertex $v (\neq v', v'')$, there exist at least $\lfloor \frac{1}{2}(d_T(v) - 1) \rfloor$ pendant edges incident with v.*

(2) *T has exactly two even-degree vertices and for every non-pendant vertex v, there exist at least $\lfloor \frac{1}{2}(d_T(v) - 1) \rfloor$ pendant edges incident with v.*

(3) *T has exactly four even-degree vertices u_1, u_2, u_3, u_4; the subgraph induced by u_1, u_2, u_3, u_4 is a path in which u_i is adjacent to u_{i+1} for $i = 1, 2, 3$ and $d_T(u_2) = d_T(u_3) = 2$; for every non-pendant vertex v, there exist at least $\lfloor \frac{1}{2}(d_T(v) - 1) \rfloor$ pendant edges incident with v.*

Theorem 4. *Let T be a tree. Then $\gamma_s'(T) = 4$ if and only if one of the following conditions is satisfied:*

(1) *T has exactly three even-degree vertices and for every non-pendant vertex v, there exist at least $\lfloor \frac{1}{2}(d_T(v) - 1) \rfloor$ pendant edges incident with v.*

(2) *T has exactly one even-degree vertex; there exist two adjacent vertices v' and v'' with $d_T(v') \geq 3$ and $d_T(v'') \geq 3$ such that v' is incident with exactly $\lfloor \frac{1}{2}(d_T(v') - 2) \rfloor$ pendant edges and v'' is incident with at least $\lfloor \frac{1}{2}(d_T(v'') - 2) \rfloor$ pendant edges; for every non-pendant vertex $v (\neq v', v'')$, there exist at least $\lfloor \frac{1}{2}(d_T(v) - 1) \rfloor$ pendant edges incident with v.*

(3) *T has exactly five even-degree vertices u_1, u_2, u_3, u_4, u_5; the subgraph induced by u_1, u_2, u_3, u_4 is a path in which u_i is adjacent to u_{i+1} for $i = 1, 2, 3$ and $d_T(u_2) = d_T(u_3) = 2$; for every non-pendant vertex v, there exist at least $\lfloor \frac{1}{2}(d_T(v) - 1) \rfloor$ pendant edges incident with v.*

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